ON THE EXISTENCE OF CARTER SUBGROUPS

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In the paper we obtain the existence criterion of a Carter subgroup in a finite group in terms of its normal series. An example showing that the criterion cannot be reformulated in terms of composition factors is given.

1 Introduction

Recall that a nilpotent self-normalizing subgroup of a group G is called a *Carter subgroup*. The classical result by Carter [1] states that every finite solvable group contains Carter subgroups and all of them are conjugate. A finite group G is said to satisfy condition **(C)** if, for every its nonabelian composition factor S and for every its nilpotent subgroup N, Carter subgroups (if exist) of $\langle \mathrm{Aut}_N(S), S\rangle$ are conjugate (the definition of $\mathrm{Aut}_N(S)$ is given below). In the recent paper [3, Theorem 10.1] it is proven that in every almost simple group with known simple socle Carter subgroups are conjugate. Thus, modulo the classification of finite simple groups, in every finite group Carter subgroups are conjugate. In the paper by a finite group we almost mean a finite group satisfying **(C)**, thus the results of the paper does not depend on the classification of finite simple groups. There exist finite groups without Carter subgroups, the minimal example is Alt_5. In the paper we give a criterion of existence of Carter subgroups in terms of normal series.

If G is a group, A, B, H are subgroups of G and B is normal in A ($B \trianglelefteq A$), then $N_H(A/B) = N_H(A) \cap N_H(B)$. If $x \in N_H(A/B)$, then x induces an automorphism $Ba \mapsto Bx^{-1}ax$ of A/B. Thus, there is a homomorphism of $N_H(A/B)$ into $\mathrm{Aut}(A/B)$. The image of this homomorphism is denoted by $\mathrm{Aut}_H(A/B)$ while its kernel is denoted by $C_H(A/B)$. In particular, if S is a composition factor of G, then for any $H \leq G$ the group $\mathrm{Aut}_H(S)$ is defined.

Let $G = G_0 \geq G_1 \geq \ldots \geq G_n = \{e\}$ be a chief series of G (recall that G is assumed to satisfy **(C)**). Then $G_i/G_{i+1} = T_{i,1} \times \ldots \times T_{i,k_i}$, where $T_{i,1} \simeq \ldots \simeq T_{i,k_i} \simeq T_i$ and T_i is a simple group. If $i \geqslant 1$, then denote by $\overline{K}_i$ a Carter subgroup of G/G_i (if it exists) and by K_i its complete preimage in G/G_{i+1}. If $i = 0$, then $\overline{K}_0 = \{e\}$ and $K_0 = G/G_1$. We say that a finite group G satisfies condition **(E)**, if, for every i, j, either $\overline{K}_i$ does not exist, or $\mathrm{Aut}_{K_i}(T_{i,j})$ contains a Carter subgroup.

The following lemma shows that the homomorphic image of a Carter subgroup is a Carter subgroup. We shall use this fact substantially.

Lemma 1. [2, Lemma 4] *Let G be a finite group, H be a normal subgroup of G satisfying* **(C)**, *and K be a Carter subgroup of G. Then KH/H is a Carter subgroup of G/H.*

Proof. By [2], Carter subgroups of KH are conjugate. Assume that there exists $x \in N_G(KH)$. Then K^x is a Carter subgroup of KH. Since Carter subgroups of KH are conjugate, there exists $y \in KH$ such that $K^x = K^{y^{-1}}$, so $xy \in N_G(K)$. Since K is a Carter subgroup of G, we obtain that $xy \in K$ and $x \in K \leq KH$, a contradiction. □

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2 Criterion

Lemma 2. *Let G be a finite group, H be a normal subgroup of G and S be a composition factor of G/H (hence of G as well).*

Then $\mathrm{Aut}_G(S) = \mathrm{Aut}_{G/H}(S)$.

Proof. Note that there exists a surjective homomorphism $\varphi : \mathrm{Aut}_G(S) \to \mathrm{Aut}_{\overline{G}}(S)$, defined by

$$\mathrm{Aut}_G(S) = (N_G(A) \cap N_G(B))\,/C_G(A/B) \to (N_G(A) \cap N_G(B))\,/\,(HC_G(A/B)) = \mathrm{Aut}_{\overline{G}}(S).$$

Since $F^*(\mathrm{Aut}_G(S)) = S \cap \mathrm{Ker}(\varphi) = \{e\}$ it folows that $\mathrm{Ker}(\varphi) = \{e\}$. $\square$

Below we shall need to know some additional information about the structure of Carter subgroups in groups of special type. Let A' be a group with a normal subgroup T'. Consider the direct product $A_1 \times \ldots \times A_k$, where $A_1 \simeq \ldots \simeq A_k \simeq A'$ and its normal subgroup $T = T_1 \times \ldots \times T_k$, where $T_1 \simeq \ldots \simeq T_k \simeq T'$. Consider the symmetric group Sym_k, acting on $A_1 \times \ldots \times A_k$ by $A_i^s = A_{i^s}$ for all $s \in S$ and define X to be a semidirect product $(A_1 \times \ldots \times A_k) \leftthreetimes \mathrm{Sym}_k$ (permutation wreath product of A' and Sym_k). Denote by A the direct product $A_1 \times \ldots \times A_k$ and by π_i the projection $\pi_i : A \to A_i$. In the introduced notations the following lemma holds.

Lemma 3. *Let G be a subgroup of X such that $T \leq G$, $G/(G \cap T)$ is nilpotent and $(G \cap A)^{\pi_i} = A_i$. Assume also that A is solvable. Let K be a Carter subgroup of G.*

Then $(K \cap A)^{\pi_i}$ is a Carter subgroup of A_i.

Proof. Assume that the statement is not true and let G be a counterexample of minimal order with minimal k. Then $S = G/(G \cap A)$ is transitive and primitive. Indeed, if S is not transitive, then $S \leq \mathrm{Sym}_{k_1} \times \mathrm{Sym}_{k-k_1}$, hence $G \leq G_1 \times G_2$. If we denote by $\psi_i : G \to G_i$ the natural homomorphism, then $G^{\psi_i} = G_i$ satisfies conditions of the lemma and $K^{\psi_i} = K_i$ is a Carter subgroup of G_i. Clearly $(G \cap A)^{\pi_j} = (G_i \cap A^{\psi_i})^{\pi_j}$, where $i = 1$ if $j \in \{1, \ldots, k_1\}$ and $i = 2$ if $j \in \{k_1 + 1, \ldots, k\}$, thus we obtain the statement by induction. If S is transitive, but is not primitive, let $\Omega_1 = \{T_1, \ldots, T_m\}, \Omega_2 = \{T_{m+1}, \ldots, T_{2m}\}, \ldots, \Omega_l = \{T_{(l-1)m+1}, \ldots, T_{lm}\}$ be a system of imprimitivity. Then it contains a nontrivial intransitive normal subgroup

$$F' \leq \underbrace{\mathrm{Sym}_m \times \ldots \times \mathrm{Sym}_m}_{l \text{ times}},$$

where $k = m \cdot l$. Consider the complete preimage F of F' in X. Then $G \cap F \leq F_1 \times \ldots \times F_l$. Denote by $\psi_i : F \to F_i$ the natural projection, then $(G \cap F)^{\psi_i} = F_i$. Note that all of F_i satisfy conditions of the lemma and, if we define $T_i' = T_{(i-1)m+1} \times \ldots \times T_{im}$, then G satisfies conditions of the lemma with $T' = T_1' \times \ldots \times T_l'$ and $A' = F$. By induction we have that $(K \cap F)^{\psi_i}$ is a Carter subgroup of F_i and, if $j \in \{m \cdot (i-1) + 1, \ldots m \cdot i\}$, then $\left((K \cap F)^{\psi_i} \cap A^{\psi_i}\right)^{\pi_j}$ is a Carter subgroup of A_j. Since $(G \cap A)^{\pi_j} = \left((K \cap F)^{\psi_i} \cap A^{\psi_i}\right)^{\pi_j}$ (for suitable i), we obtain the statement by induction.

Let Y' be a minimal normal subgroup of G contained in T (if Y' is trivial, then T is trivial and we have nothing to prove, since G is nilpotent in this case). Thus Y' is a normal elementary Abelian p-group. Let $Y_i = (Y')^{\pi_i}$, then $Y = Y_1 \times \ldots \times Y_k$ is a nontrivial normal subgroup of G (Y is a subgroup of G since $T \leq G$). Let $\bar{\pi}_i : (G \cap A) \to A_i/Y_i = \overline{A}_i$ be the projection corresponding to π_i. Denote by $\overline{K} = KY/Y$ the corresponding Carter subgroup of $\overline{G} = G/Y$. Then $\overline{G}$ satisfies conditions of the lemma. By induction, $(\overline{K} \cap \overline{A})^{\bar{\pi}_i}$ is a Carter subgroup of

$\overline{A}_i$. Let K_1 be a complete preimage of $\overline{K}$ in G and let Q be a Hall p'-subgroup of K_1. Then $(Q\cap A)^{\pi_i}$ is a Hall p'-subgroup of $(K_1\cap A)^{\pi_i}$. In view of the proof of [4, Theorem 20.1.4], we obtain that $K=N_{K_1}(Q)$ is a Carter subgroup of G and $(N_{K_1\cap A}(Q\cap A))^{\pi_i}$ is a Carter subgroup of A_i. Thus we need to show that $(N_{K_1\cap A}(Q\cap A))^{\pi_i}=(N_{K_1\cap S}(Q))^{\pi_i}$. By induction, the equality $(N_{\overline{K}\cap\overline{A}}(\overline{A}\cap\overline{Q}))^{\bar{\pi}_i}=(N_{\overline{K}\cap\overline{G}}(\overline{Q}))^{\bar{\pi}_i}$ holds. Thus we need to prove that $(N_Y(Q\cap A))^{\pi_i}=(N_Y(Q))^{\pi_i}$. Note also that $(N_Y(Q\cap A))^{\pi_i}\leq N_{Y_i}((Q\cap A)^{\pi_i})$.

Since S is a transitive and primitive nilpotent subgroup of Sym_k, then $k=r$ is prime and $S=\langle s\rangle$ is cyclic. If $r=p$, then $Q\cap A=Q$ and we have nothing to prove. Otherwise let h be an r-element of K, generating S modulo $K\cap A$. Clearly $Q=(Q\cap A)\langle h\rangle$. Let $t\in Y_i$ be an element of $N_{Y_i}((Q\cap A)^{\pi_i})$. Then $(t\cdot t^h\cdot\ldots\cdot t^{h^{r-1}})\in N_Y(Q)$ and $t^{\pi_i}=(t\cdot t^h\cdot\ldots\cdot t^{h^{r-1}})^{\pi_i}$, hence $(N_Y(Q\cap A))^{\pi_i}\leq N_{Y_i}((Q\cap A)^{\pi_i})\leq(N_Y(Q))^{\pi_i}\leq(N_Y(Q\cap A))^{\pi_i}$. $\square$

Theorem. *Let G be a finite group. Then G contains a Carter subgroup if and only if G satisfies* **(E)**.

Proof. We prove first the part "only if". Let H be a minimal normal subgroup of G. Then $H=T_1\times\ldots\times T_k$, where $T_1\simeq\ldots\simeq T_k\simeq T$ are simple groups.

If H is elementary Abelian (i. e., T is cyclic of prime order), then $\mathrm{Aut}(T)$ is solvable and contains a Carter subgroup. Assume that T is a nonabelian simple group. Clearly K is a Carter subgroup of KH. By [2, Lemma 3] we obtain that $\mathrm{Aut}_{KH}(T_i)$ contains a Carter subgroup for all i.

Now we prove the part "if". Again assume by contradiction that G is a counterexample of minimal order, i. e., that G does not contain a Carter subgroup, but, G satisfies **(E)**. Let H be a minimal normal subgroup of G. Then $H=T_1\times\ldots\times T_k$, where $T_1\simeq\ldots\simeq T_k\simeq T$, and T is a finite simple group.

By definition G/H satisfies **(E)**, thus, by induction, there exists a Carter subgroup $\overline{K}$ of $\overline{G}=G/H$. Let K be a complete preimage of $\overline{K}$, then K satisfies **(E)**. If $K\neq G$, then, by induction K contains a Carter subgroup K'. Note that K' is a Carter subgroup of G. Indeed, assume that $x\in N_G(K')\setminus K'$. Since $K'H/H=\overline{K}$ is a Carter subgroup of $\overline{G}$, we have that $x\in K$. But K' is a Carter subgroup of K, thus $x\in K'$. Hence $G=K$, i. e. G/H is nilpotent.

If H is Abelian, then G is solvable, therefore G contains a Carter subgroup. So assume that T is a nonabelian finite simple group. We first show that $C_G(H)$ is trivial. Assume that $C_G(H)=M$ is nontrivial. Since T is a nonabelian simple group, it follows that $M\cap H=\{e\}$, so M is nilpotent. By Lemma 2 we have that G/M satisfy **(E)**. By induction we obtain that G/M contains a Carter subgroup $\overline{K}$. Let K' be a complete preimage of $\overline{K}$ in G. Then K' is solvable, hence contains a Carter subgroup K. Like above we obtain that K is a Carter subgroup of G, a contradiction. Hence $C_G(H)=\{e\}$.

Since H is a minimal normal subgroup of G, we obtain that $\mathrm{Aut}_G(T_1)\simeq\mathrm{Aut}_G(T_2)\simeq\ldots\simeq\mathrm{Aut}_G(T_k)$. Thus there exists a monomorphism

$$\varphi: G\to(\mathrm{Aut}_G(T_1)\times\ldots\times\mathrm{Aut}_G(T_k))\leftthreetimes\mathrm{Sym}_k=G_1$$

and we identify G with G^{φ}. Denote by K_i a Carter subgroup of $\mathrm{Aut}_G(T_i)$ and by A the subgroup $\mathrm{Aut}_G(T_1)\times\ldots\times\mathrm{Aut}_G(T_k)$. Since G/H is nilpotent, then $K_iT_i=\mathrm{Aut}_G(T_i)$ and $G_1=(K_1T_1\times\ldots\times K_kT_k)\leftthreetimes\mathrm{Sym}_k$. Let $\pi_i: G\cap A\to(G\cap A)/C_{(G\cap A)}(T_i)$ be the canonical projections. Since $G/(G\cap A)$ is transitive, we obtain that $(G\cap A)^{\pi_i}=K_iT_i$.

Since $\mathrm{Aut}_{G\cap A}(T_i)=K_iT_i$, hence $G\cap A$ satisfies **(E)**. By induction it contains a Carter subgroup M. By [2, Lemma 3] we obtain that M^{π_i} is a Carter subgroup of K_iT_i, therefore we

may assume $M^{\pi_i} = K_i$. In particular, if $R = (K_1 \cap T_1) \times \ldots \times (K_k \cap T_k)$, then $M \leq N_G(R)$. In view of [3], Carter subgroups in every finite group are conjugate. Since $(G \cap A)/H$ is nilpotent we obtain that $G \cap A = MH$, hence $G = N_G(M)H$. Moreover $N_G(M) \cap A = M$, so $N_G(M)$ is solvable. Since M normalizes R, $M^{\pi_i} = K_i$, we obtain that $N_G(M)$ normalizes R, hence $N_G(M)R$ is solvable. Therefore it contains a Carter subgroup K. By Lemma 3, $(K \cap A)^{\pi_i}$ is a Carter subgroup of $(N_G(M)R \cap A)^{\pi_i}$ (R plays the role of T from Lemma 3 in this case), so $(K \cap A)^{\pi_i} = K_i$. Assume that $x \in N_G(K) \setminus K$. Since $G/H = N_G(M)H/H = KH/H$ it follows that $x \in H$. Therefore $x^{\pi_i} \in (N_G(K) \cap A)^{\pi_i} \leq N_{T_i}((K \cap A)^{\pi_i}) = K_i$. Since $\bigcap_i \mathrm{Ker}(\pi_i) = \{e\}$, it follows that $x \in R \leq N_G(M)R$. But K is a Carter subgroup of $N_G(M)R$, hence $x \in K$. This contradiction completes the proof. $\square$

3 Example

In this section we construct an example showing, that we can not substitute condition **(E)** by a weaker condition: for every composition factor S of G, $\mathrm{Aut}_G(S)$ contains a Carter subgroup. This example also shows that an extension of a group containing a Carter subgroup by a group containing a Carter subgroup may fail to contain a Carter subgroup.

Consider $L = \Gamma SL_2(3^3) = PSL_2(3^3) \leftthreetimes \langle \varphi \rangle$, where φ is a field automorphism of $PSL_2(3^3)$. Let $X = (L_1 \times L_2) \leftthreetimes \mathrm{Sym}_2$, where $L_1 \simeq L_2 \simeq L$ and if $\sigma = (1,2) \in \mathrm{Sym}_2 \setminus \{e\}$, $(x,y) \in L_1 \times L_2$, then $\sigma(x,y)\sigma = (y,x)$ (permutation wreath product of L and Sym_2). Denote by $H = PSL_2(3^3) \times PSL_2(3^3)$ the minimal normal subgroup of X and by $M = L_1 \times L_2$. Let $G = (H \leftthreetimes \langle (\varphi, \varphi^{-1}) \rangle) \leftthreetimes \mathrm{Sym}_2$ be a subgroup of X. Then the following statements hold:

1. For every composition factor S of G, $\mathrm{Aut}_G(S)$ contains a Carter subgroup.

2. $G \cap M \trianglelefteq G$ contains a Carter subgroup.

3. $G/(G \cap M)$ is nilpotent.

4. G does not contain a Carter subgroup.

1. Clearly we need to verify the statement for nonabelian composition factors only. Every nonabelian composition factor S of G is isomorphic to $PSL_2(3^3)$ and $\mathrm{Aut}_G(S) = L$. In view of [3, Theorem 7.1] we obtain that L contains a Carter subgroup (coinciding with a Sylow 3-subgroup).

2. Since $(G \cap M)/H$ is nilpotent and from the previous statement we obtain that $G \cap M$ satisfies **(E)**, hence contains a Carter subgroup (it is easy to see that a Sylow 3-subgroup of $G \cap M$ is a Carter subgroup of $G \cap M$).

3. Evident.

4. Assume that K is a Carter subgroup of G. Then KH/H is a Carter subgroup of G/H. But G/H is a nonabelian group of order 6, hence $G/H \simeq \mathrm{Sym}_3$ and KH/H is a Sylow 2-subgroup of G/H. In view of [2, Lemma 3] it follows that $\mathrm{Aut}_K(PSL_2(3^3))$ is a Carter subgroup of $\mathrm{Aut}_{KH}(PSL_2(3^3)) = PSL_2(3^3)$. But $PSL_2(3^3)$ does not contain Carter subgroups in view of [3, Theorem 7.1].

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